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A PROBABILISTIC APPROACH TO A THEOREM OF GILBARG AND SERRIN

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ABSTRACT
We give a probabilistic proof via coupling of an extended version of a theorem of Gilbarg and Serrin.

1. Introduction

Let

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}$$

be defined on R^d, $d \geq 2$. Except in the statement of Harnack's inequality below, we shall always assume that the coefficients satisfy: $a_{ij} \in C^\alpha(R^d)$, $b_i \in C^\alpha(R^d)$ and $a(x) = \{a_{ij}(x)\}$ is positive definite for each $x \in R^d$. The main result of this paper will be proved under

ASSUMPTION A. (i) $|b(x)| \leq M/(1+|x|)$, for all $x \in R^d$;
(ii) $\lambda |\zeta|^2 \leq \sum_{i,j=1}^d a_{ij}(x)\zeta_i\zeta_j \leq \Lambda |\zeta|^2$, for all $x, \zeta \in R^d$, where $0 < \lambda \leq \Lambda < \infty$.

We will give a probabilistic proof via coupling of the following Liouville-type result.

THEOREM 1. *Let L satisfy Assumption A.*
(i) *If $0 < u \in C^2(R^d)$ satisfies $Lu = 0$ in R^d, then $u =$* constant.
(ii) *Let $D \subset R^d$ be an exterior domain. If $0 < u \in C^2(D) \cap C(\bar{D})$ satisfies $Lu = 0$ in D, then $\lim_{|x| \to \infty} u(x)$ exists (but may be infinite).*

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In order to implement our coupling, we utilize the following version of Harnack's inequality proved in 1980 by Safanov ([7],[4]).

HARNACK'S INEQUALITY. Let

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}$$

be defined on a bounded domain $\Omega \subset R^d$. Assume that

$$\mu |\zeta|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \zeta_i \zeta_j \leq \Lambda |\zeta|^2, \qquad \text{for all } x \in \Omega \quad \text{and} \quad \zeta \in R^d,$$

where $0 < \lambda \leq \Lambda < \infty$, and that $\|b\|_\infty \equiv \sup_{x \in \Omega} |b(x)| < \infty$. Let $u \in W^{2,p}(\Omega)$ satisfy $Lu = 0$ and $u > 0$ in Ω. Then for $D \subset \bar{D} \subset \Omega$, $\sup_D u \leq c \inf_D u$, where $c = c(\Lambda/\lambda, \|b\|_\infty/\lambda, D, \Omega)$.

A nonprobabilistic proof of part (ii) of Theorem 1 (under the additional assumption that u satisfy a certain growth condition if $d \geq 3$) was given in 1956 by Gilbarg and Serrin [3], who relied on Harnack's inequality. They used a version of Harnack's inequality proved by Serrin [8], which is weaker than Safanov's version. In particular, for $d \geq 3$, the constant c in Serrin's version depended on the modulus of continuity of a_{ij}. Consequently, Gilbarg and Serrin's theorem required that $a_{ij}(x)$ possess a limit as $|x| \to \infty$ in the case of R^d, $d \geq 3$. They remarked that if Harnack's inequality were strengthened, then their result would be strengthened accordingly. Part (i) of Theorem 1 (under the additional condition that u be bounded if $d \geq 3$) was stated without proof as a corollary to part (ii).

Before discussing our coupling technique, we investigate the connection between the statements in part (i) and part (ii) of Theorem 1 for general L, irrespective of Assumption A. Indeed, we feel that Theorem 2 below is of interest independent of the theorem we wish to prove. In the case that L generates a transient diffusion we have the following theorem.

THEOREM 2. *Assume that L generates a transient diffusion and let D be a smooth exterior domain. Then there are no nonconstant bounded C^2-solutions of $Lu = 0$ in R^d if and only if for every bounded $u \in C^2(D) \cap C(\bar{D})$ solving $Lu = 0$ in D and for every sequence $\{x_n\}_{n=1}^\infty$ such that $\lim_{|n| \to \infty} |x_n| = \infty$ and $\lim_{n \to \infty} P_{x_n}(\tau_D < \infty) = 0$, the sequence $\{u(x_n)\}_{n=1}^\infty$ tends to a limit which is independent of $\{x_n\}_{n=1}^\infty$. In particular, if $\lim_{|x| \to \infty} P_x(\tau_D < \infty) = 0$, then there are no nonconstant bounded C^2-solutions of $Lu = 0$ in R^d if and only if every bounded solution $u \in C^2(D) \cap C(\bar{D})$ of $Lu = 0$ in D tends to a limit as $|x| \to \infty$.*

REMARK. $v(x) \equiv P_x(\tau_D < \infty) \in C^2(R^d)$ and is the smallest positive solution of $Lu = 0$ in D with $u = 1$ on ∂D.

In the case that L generates a recurrent diffusion, we have the following well-known result.

PROPOSITION 1. *Let L generate a recurrent diffusion. If $0 < u \in C^2(R^d)$ and $Lu = 0$ in R^d, then $u =$* const.

We now turn to our coupling technique. We introduce the following notation. Let $\Omega = C([0,\infty), R^d)$ denote the space of continuous functions from $[0,\infty)$ to R^d and let P_x denote the probability measure on Ω corresponding to the diffusion generated by L and starting from $x \in R^d$. Denote trajectories in Ω by $X(\cdot)$. Denote elements of the product space $\Omega \times \Omega$ by $(X(\cdot), Y(\cdot))$. We distinguish between hitting times for $X(\cdot)$ and $Y(\cdot)$ by writing $\tau_D(X(\cdot))$ and $\tau_D(Y(\cdot))$. For a probability measure Q on $\Omega \times \Omega$, let Q_i denote the ith marginal, $i = 1,2$. The following coupling result holds irrespective of Assumption A. Similar types of results have been proven elsewhere; see for example [5] and [6].

THEOREM 3. (i) *Assume that for each $(x,y) \in R^d \times R^d$ there exists a probability measure $Q^{(x,y)}$ on $\Omega \times \Omega$ such that*

(1) $Q_1^{(x,y)} = P_x$;

(2) $Q_2^{(x,y)} = P_y$;

(3) $Q^{(x,y)}\big((X(\cdot), Y(\cdot)) \in \Omega \times \Omega : \exists s_1 s_2 \geq 0$ *such that* $X(s_1 + t) = Y(s_2 + t)$, *for all* $t \geq 0\big) = 1$.

Then every bounded solution $u \in C^2(R^d)$ of $Lu = 0$ in R^d is constant.

(ii) *Let $D \subset R^d$ be an exterior domain and let L generate a recurrent diffusion. Assume that for each $\epsilon > 0$ there exists an n_ϵ such that for each $(x,y) \in R^d \times R^d$ with $|x| \geq n_\epsilon$ and $|y| \geq n_\epsilon$, there exists a probability measure $Q^{(x,y)}$ on $\Omega \times \Omega$ satisfying*

(1) $Q_1^{(x,y)} = P_x$;

(2) $Q_2^{(x,y)} = Py$;

(3) $Q^{(x,y)}\big(X(\tau_D(X(\cdot))) = Y(\tau_D(Y(\cdot)))\big) \geq 1 - \epsilon$.

Then $\lim_{|x|\to\infty} u(x)$ exists for every bounded solution $u \in C^2(D) \cap C(\bar{D})$ of $Lu = 0$ in D.

Under Assumption A, the coupling of Theorem 3 can be implemented. We have

THEOREM 4. *Let L satisfy Assumption A. Then*

(i) *the coupling in* (3)(i) *may be achieved,*

(ii) *the coupling in* (3)(ii) *may be achieved.*

The proof of Theorem 4 involves a scaling argument which can be used to give an easy proof of the following proposition.

Proposition 2. *Let L satisfy Assumption A.*

(i) *Let L generate a recurrent diffusion and let D be an exterior domain. Then* $v(x) = P_x(\tau_D < \infty)$ *satisfies* $\lim_{|x|\to\infty} v(x) = 0$.

(ii) *Let L generate a recurrent diffusion, let D be an exterior domain and let* $0 < u \in C^2(D) \cap C(\bar{D})$ *satisfy* $Lu = 0$ *in D. If u is unbounded, then* $\lim_{|x|\to\infty} u(x) = \infty$.

(iii) *Let L generate a transient diffusion and let D be an exterior domain. If* $0 < u \in C^2(R^d)$ *satisfies* $Lu = 0$ *in* R^d *or if* $0 < u \in C^2(D)$ *satisfies* $Lu = 0$ *in D, then u is bounded.*

Theorem 1 now follows from Theorems 2, 3, and 4 and Propositions 1 and 2. In Section 2 we will prove Theorems 2 and 3 and, for the sake of completeness, Proposition 1. The proofs of Theorem 4 and Proposition 2 are given in Section 3.

2. Proof of Proposition 1 and Theorems 2 and 3

Proof of Proposition 1. Assume to the contrary that u is not constant. Then there exist bounded open sets D_1 and D_2 such that $\sup_{x\in D_1} u(x) < \inf_{x\in D_2} u(x)$. Let $\tau_{D_2} = \inf\{t > 0 : X(t) \in D_2\}$ and $\tau_n = \inf\{t > 0 : |X(t)| = n\}$. Since $Lu = 0$, $u(X(t \wedge \tau_{D_2} \wedge \tau_n))$ is a P_x-martingale for each $x \in R^d$. Let $x_0 \in D_1$. Then $u(x_0) = E_{x_0}u(X(t \wedge \tau_{D_2} \wedge \tau_n))$. Letting $t \to \infty$ gives $u(x_0) = E_{x_0}u(X(\tau_{D_2} \wedge \tau_n))$. By assumption, $X(t)$ is a recurrent process so $P_{x_0}(\tau_{D_2} < \infty) = 1$. Thus letting $n \to \infty$ and using the positivity of u, we obtain the contradiction $u(x_0) \ge E_{x_0}u(X(\tau_{D_2})) \ge \inf_{x\in D_2} u(x)$.

Proof of Theorem 2. First assume that for every bounded solution $u \in C^2(D) \cap C(\bar{D})$ of $Lu = 0$ in the exterior domain D and for every sequence $\{x_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} |x_n| = \infty$ and $\lim_{n\to\infty} P_{x_n}(\tau_D < \infty) = 0$, the sequence $\{u(x_n)\}_{n=1}^\infty$ tends to a limit which is independent of $\{x_n\}_{n=1}^\infty$. Let $u \in C^2(R^d)$ be bounded and satisfy $Lu = 0$ in R^d. We will prove that u is constant. Since u is also a solution of the exterior problem, it follows from our assumption that $c \equiv \lim_{n\to\infty} u(x_n)$ exists for any sequence $\{x_n\}_{n=1}^\infty$, such that $\lim_{n\to\infty} |x_n| = \infty$ and $\lim_{n\to\infty} P_{x_n}(\tau_D < \infty) = 0$. Now $v(x) = P_x(\tau_D < \infty)$ satisfies

$$\begin{aligned}\lim_{t\to\infty} v(X(t)) &= \lim_{t\to\infty} P_x\big(|X(s) \notin D \text{ for some } s \ge t \,|\, X(r),\ 0 \le r \le t\big) \\ &= 1_{\{|X(t_n)\notin D \text{ for some sequence } t_n\to\infty\}} \text{ a.s. } P_x\end{aligned}$$

(see [1, Theorem 9.5.1]). Thus, by transience, $\lim_{t\to\infty} v(X(t)) = 0$ a.s. P_x and, consequently, $\lim_{t\to\infty} u(x(t))) = c$ a.s. P_x. Since $Lu = 0$, $u(X(t))$ is a bounded P_x-martingale, and we obtain $u(x) = E_x u(X(t)) = \lim_{t\to\infty} E_x u(X(t)) = c$, for all $x \in R^d$. Thus u is constant.

Conversely, assume that every bounded solution $u \in C^2(R^d)$ of $Lu = 0$ in R^d is constant. Let $\{x_n\}_{n=1}^{\infty}$ satisfy $\lim_{n\to\infty} |x_n| = \infty$ and $\lim_{n\to\infty} P_{x_n}(\tau_D < \infty) = 0$ and let $u \in C^2(D) \cap C(D)$ be a bounded solution of $Lu = 0$ in the exterior region D. We will show that $u(x_n)$ possesses a limit at infinity and that the limit is independent of $\{x_n\}_{n=1}^{\infty}$. First extend u in an arbitrary manner so that it is defined on all of R^d. Define

$$\mathcal{U}(X(\cdot)) = \begin{cases} \lim_{t\to\infty} u(X(t)) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $Lu = 0$ in D, $u(X(t \wedge \tau_D))$ is a bounded P_x-martingale for $x \in D$ and thus it converges a.s. P_x for $x \in D$. From this and the strong Markov property, it follows that $|\mathcal{U}| \le \sup_{y\in D} |u(y)|$ a.s. P_x for $x \in R^d$. Clearly $\mathcal{U}$ is measurable with respect to the invariant σ-field and thus $h(x) = E_x \mathcal{U}$ is bounded and L-harmonic [2]. By our assumption, then, $h(x) \equiv c$. Now define

$$\mathcal{V}(X(\cdot)) = \begin{cases} \lim_{t\to\infty} u(X(t \wedge \tau_D)) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $u(X(t \wedge \tau_D))$ converges a.s. P_x, for $x \in D$, $|\mathcal{V}| \le \sup_{x\in D} |u(x)|$ a.s. P_x for $x \in D$. Furthermore,

$$u(x) = E_x \mathcal{V} = E_x \mathcal{V} \mathcal{I}_{(\tau_D = \infty)} + E_x \mathcal{V} \mathcal{I}_{(\tau_D < \infty)}, \qquad \text{for } x \in D. \tag{2.1}$$

But $\mathcal{V}\mathcal{I}_{(\tau_D=\infty)} = \mathcal{U}\mathcal{I}_{(\tau_D=\infty)}$. Thus, we have

$$c = h(x) = E_x \mathcal{U} = E_x \mathcal{V} \mathcal{I}_{\tau_D=\infty} + E_x \mathcal{U} \mathcal{I}_{(\tau_D<\infty)}.$$

Plugging this into (2.1) gives

$$u(x) = c - E_x \mathcal{U} \mathcal{I}_{(\tau_D<\infty)} + E_x \mathcal{V} \mathcal{I}_{(\tau_D<\infty)}. \tag{2.2}$$

Since $|\mathcal{U}|$ and $|\mathcal{V}|$ are bounded by $\sup_{x\in D} |u(x)|$ a.s. P_x for all $x \in D$, and since by assumption, $\lim_{n\to\infty} P_{x_n}(\tau_D < \infty) = 0$, it follows from (2.2) that $\lim_{n\to\infty} u(x_n) = c$. This completes the proof of the theorem.

We now turn to the

PROOF OF THEOREM 3. We will denote expectations with respect to P_x by E_x and expectations with respect to $Q^{x,y}$ by $E^{Q^{x,y}}$.

Part (i). Let $u \in C^2(R^d)$ be a bounded solution of $Lu = 0$ in R^d. Then $u(X(t))$ is a bounded P_x-martingale and thus converges a.s. P_x. By the same token, using the notation $Y(\cdot)$ for paths in Ω, it follows that $u(Y(t))$ converges a.s. P_y. Thus, by properties (1) and (2) of $Q^{(x,y)}$, it follows that

$$(u(X(t)),u(Y(t))) \text{ converges as } t \to \infty \text{ a.s. } Q^{(x,y)}, \tag{2.3}$$

$$u(x) = \lim_{t\to\infty} E_x u(X(t)) = \lim_{t\to\infty} E^{Q^{(x,y)}} u(X(t)), \tag{2.4}$$

$$u(y) = \lim_{t\to\infty} E_y u(Y(t)) = \lim_{t\to\infty} E^{Q^{(x,y)}} u(Y(t)). \tag{2.5}$$

From (2.3), (2.4) and (2.5) and property (3) of $Q^{(x,y)}$, we conclude that $u(x) = u(y)$. As x and y are arbitrary, $u =$ constant.

Part (ii). Fix $\epsilon > 0$ and let $x,y \in D$ satisfy $|x| > n_\epsilon$ and $|y| > n_\epsilon$. By properties (1) and (2) of $Q^{(x,y)}$, we have

$$\mu_x(dz) = P_x(X(\tau_D) \in dz) = Q^{(x,y)}(X(\tau_D(X(\cdot))) \in dz)$$

and

$$\mu_y(dz) = P_y(X(\tau_D) \in dz) = Q^{(x,y)}(Y(\tau_D(Y(\cdot))) \in dz).$$

Since $Lu = 0$ in D, $u(X(t \wedge \tau_D))$ is a bounded P_x-martingale and thus $u(x) = E_x u(X(t \wedge \tau_D))$. Letting $t \to \infty$ and using the boundedness of u and the recurrence of $X(t)$ gives $u(x) = E_x u(X(\tau_D)) = \int_{\partial D} u(z)\mu_x(dz)$. Similarly, $u(y) = \int_{\partial D} u(z)\mu_y(dz)$. By property (3), it follows that

$$|u(x) - u(y)| = \left| \int_{\partial D} u(z)\mu_x(dz) - \int_{\partial D} u(z)\mu_y(dz) \right| \le 2\epsilon \sup_{z\in\partial D} |u(z)|.$$

3. Proofs of Theorem 4 and Proposition 2

PROOF OF THEOREM 4, *part (i).* Let $m = (\max(|x_0|,|y_0|) + \gamma) V 2$, for some $\gamma > 0$. For $X(\cdot) \in \Omega$, define the following stopping times.

$$\sigma_0 = \sigma_0(X(\cdot)) = \inf\{t \ge 0 : |X(t)| = m\},$$

$$\sigma_j = \sigma_j(X(\cdot)) = \inf\{t > \sigma_{j-1} : |X(t)| = m^{j+1}\}, \quad j = 1,2,\ldots .$$

It will also be convenient to define the following stopping times:

$$\tau_j = \tau_j(X(\cdot)) = \inf\{t \geq 0 : |X(t)| = m^{j+1}\}, \quad j = 0,1,2,\ldots .$$

Define $\Sigma_j = \{|x| = m^{j+1}\}$, $j = 0,1,2,\ldots$. For $j = 0,1,2,\ldots$ and $|x| \leq m^{j+1}$, define the harmonic measure $\mu_x^j(dz) \in \mathcal{P}(\Sigma_j)$ by $\mu_x^j(dz) = P_x(X(\tau_j) \in dz)$.

Our coupling turns on the following

Claim. There exists a constant $0 < c < 1$ and for each $j = 0,1,2,\ldots$ a probability measure $\nu^j \in \mathcal{P}(\Sigma_j)$ such that

$$\mu_x^0(dz) \geq c\nu^0(dz), \qquad \text{for } x = x_0 \text{ or } y_0$$

and

$$\mu_x^j(dz) \geq c\nu^j(dz), \qquad \text{for } |x| = m^j, \quad j = 1,2,\ldots .$$

We will now use this result to prove the theorem and then return to prove the Claim.

For $(X(\cdot), Y(\cdot)) \in \Omega \times \Omega$, we will let t denote the time variable in $X(\cdot)$ and s denote the time variable in $Y(\cdot)$. Let $X(t)$ and $Y(s)$ run independently starting from x_0 and y_0, respectively, up until time $t = \sigma_0(X(\cdot))$ and $s = \sigma_0(Y(\cdot))$. That is, define Q on $\mathcal{F}_{\sigma_0(X(\cdot))} \times \mathcal{F}_{\sigma_0(Y(\cdot))}$ by $P_{x_0} \times P_{y_0}$.

The distribution of $X(\sigma_0(X(\cdot)))$ under Q or, equivalently, under P_{x_0}, and the distribution of $Y(\sigma_0(Y(\cdot)))$ under Q or, equivalently, under P_{y_0}, are respectively $\mu_{x_0}^0(dz)$ and $\mu_{y_0}^0(dz)$. By our Claim, we may write each of these as a convex combination of probability measures as follows:

$$\mu_{x_0}^0(dz) = c\nu^0(dz) + (1-c)\left[\frac{(\mu_{x_0} - c\nu)(dz)}{1-c}\right],$$

$$\mu_{y_0}^0(dz) = c\nu^0(dz) + (1-c)\left[\frac{(\mu_{y_0} - c\nu)(dz)}{1-c}\right].$$

We now describe how to continue the definition of Q starting from $t = \sigma_0(X(\cdot))$ and $s = \sigma_0(Y(\cdot))$. With probability c, start both $X(\cdot)$ and $Y(\cdot)$ from the distribution $\nu^0(dz)$ and couple them, that is, run them so that

$$X(\sigma_0(X(\cdot)) + r) = Y(\sigma_0(Y(\cdot)) + r) \qquad \text{for all } r \geq 0$$

and so that $X(\sigma_0(X(\cdot)) + \cdot\,) \sim P_{\nu^0}$. With probability $1-c$, let $X(\cdot)$ and $Y(\cdot)$ continue to run independently with $X(\cdot)$ starting from $X(\sigma_0(X(\cdot)))$ and $Y(\cdot)$ starting from $Y(\sigma_0(Y(\cdot)))$ with distributions

$$\frac{(\mu_{x_0} - c\nu)(dz)}{1-c} \quad \text{and} \quad \frac{(\mu_{y_0} - c\nu)(dz)}{1-c}$$

respectively. Use this recipe to continue the definition of Q from

$$\mathfrak{F}_{\sigma_0(X(\cdot))} \times \mathfrak{F}_{\sigma_0(Y(\cdot))} \quad \text{up to} \quad \mathfrak{F}_{\sigma_1(X(\cdot))} \times \mathfrak{F}_{\sigma_1(Y(\cdot))}.$$

To extend the definition of Q from

$$\mathfrak{F}_{\sigma_1(X(\cdot))} \times \mathfrak{F}_{\sigma_1(Y(\cdot))} \quad \text{up to} \quad \mathfrak{F}_{\sigma_2(X(\cdot))} \times \mathfrak{F}_{\sigma_2(Y(\cdot))}$$

we proceed as follows. By time $t = \sigma_1(X(\cdot))$ and $s = \sigma_1(Y(\cdot))$, $X(\cdot)$ and $Y(\cdot)$ are already coupled with probability c; continue to run them coupled. With probability $1-c$, $X(\cdot)$ and $Y(\cdot)$ are still running independently at time $t = \sigma_1(X(\cdot))$ and $s = \sigma_1(Y(\cdot))$. Now the distributions of $X(\sigma_1(X(\cdot)))$ and $Y(\sigma_1(Y(\cdot)))$, given that they are still running independently, are respectively

$$\hat{\mu}^1_{x_0}(dz) \equiv \int_{\Sigma_0} \mu^1_y(dz) \, \frac{(\mu_{x_0} - c\nu)(dy)}{1-c}$$

and

$$\hat{\mu}^1_{y_0}(dz) \equiv \int_{\Sigma_0} \mu^1_y(dz) \, \frac{(\mu_{y_0} - c\nu)(dy)}{1-c}.$$

By the Claim, $\mu^1_y(dz) \geq c\nu^1(dz)$, for all $y \in \Sigma_0$. Thus we may write $\hat{\mu}^1_{x_0}$ and $\hat{\mu}^1_{y_0}$ as convex combinations of probability distributions as follows:

$$\hat{\mu}^1_{x_0}(dz) = c\nu^1(dz) + (1-c)\left[\frac{(\hat{\mu}^1_{x_0} - c\nu^1)(dz)}{1-c}\right],$$

$$\hat{\mu}^1_{y_0}(dz) = c\nu^1(dz) + (1-c)\left[\frac{(\hat{\mu}^1_{y_0} - c\nu^1)(dz)}{1-c}\right].$$

Thus, as before, with probability c we couple and with probability $1-c$ we allow the paths to continue independently up till time $t = \sigma_2(X(\cdot))$ and $s = \sigma_2(Y(\cdot))$. Continuing inductively, by time $t = \sigma^k(X(\cdot))$ and $s = \sigma^k(Y(\cdot))$, the probability that paths are coupled is $\sum_{j=0}^k c(1-c)^j$ and as $k \to \infty$, this probability converges to one. Clearly, the three properties of Theorem 3(i) are met.

We now prove the Claim for $j \geq 1$. The same argument works for $j = 0$ but the notation must be altered to accommodate this case. Fix $j \geq 1$ and consider $X(t)$, $0 \leq t \leq \tau_j$ under $P_x|_{\mathfrak{F}_0^{\tau_j}}$ with $|x| = m^j$. Scale the process by a factor of $1/m^j$. The new generator is $(1/m^{2j})\hat{L}_j$, where

$$\hat{L}_j = \frac{1}{2}\sum_{i,k=1}^{d} a_{ik}^{(j)}(x)\,\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{d} b_i^{(j)}(x)\,\frac{\partial}{\partial x_i},$$

$$a^{(j)}(x) = a(m^j x) \quad \text{and} \quad b^{(j)}(x) = m^j b(m^j x).$$

If we let $\hat{P}_z^j$ denote the probability measure on $C([0,\infty), R^d)$ corresponding to the generator $(1/m^{2j})\hat{L}_j$ and starting from $z \in R^d$, then the measure induced by $P_x|_{\mathcal{F}_0^{\tau_j}}$ under the scaling is $(\hat{P}_{x/m^j}^j)|_{\mathcal{F}^{\tau_0}}$. Let $\hat{\mu}_y^j(dz) = \hat{P}_y^j(X(\tau_0) \in dz)$, for $|y| \le m$. With a slight abuse of notation, let $\mu_x^j(d\phi)$ and $\hat{\mu}_y^j(d\phi)$ denote the marginal distributions of $\mu_x^j(dz)$ and $\hat{\mu}_y^j(dz)$ on S^{d-1}. Then, clearly,

$$\hat{\mu}_{x/m^j}^j(d\phi) = \mu_x^j(d\phi), \qquad \text{for } |x| \le m^{j+1}.$$

Thus to prove the Claim, it suffices to find a $\nu^j(d\phi) \in \mathcal{P}(S^{d-1})$ and $c > 0$ such that $\hat{\mu}_y^j(d\phi) > c\nu^j(d\phi)$, for all $j = 1,2,\ldots$ and $|y| = 1$. We will show that there exists a $c > 0$ such that for any fixed y_0 with $|y_0| = 1$,

$$\text{(3.1)} \qquad \hat{\mu}_y^j(d\phi) \ge c\hat{\mu}_{y_0}^j(d\phi), \qquad \text{for } j = 1,2,\ldots \quad \text{and} \quad |y| = 1,$$

and thus we map pick $\nu^j(d\phi) = \hat{\mu}_{y_0}^j(d\phi)$.

To prove (3.1), it is enough to show that for all nonnegative $f \in C^\infty(S^{d-1})$, all $j = 1,2,\ldots$ and all $|y| = 1$,

$$\int_{S^{d-1}} f(\phi)\hat{\mu}_y^j(d\phi) \ge c\int_{S^{d-1}} f(\phi)\hat{\mu}_{y_0}^j(d\phi),$$

for some $0 < c < 1$. Let

$$u_j(y) = \int_{S^{d-1}} f(\phi)\hat{\mu}_y^j(d\phi), \qquad \text{for } |y| \le m.$$

By the smoothness assumptions on a_{ij} and b_i, there exists a solution $v_j \in C^{2,\alpha}$ of $\hat{L}_j v_j = 0$ in $|y| < m$ with $v_j = f$ on $|y| = m$ [4]. In fact, then, $v_j = u_j$. Now consider the domains $D = \{\frac{3}{4} < |x| < \frac{3}{2}\}$ and $\Omega = \{\frac{1}{2} < |x| < m\}$. The diffusion matrix $a^j(x)$ satisfies Assumption A(ii) for $x \in \Omega$. By Assumption A(i),

$$\sup_j \sup_{x \in \Omega} |b^{(j)}(x)| \le \sup_j \sup_{x \in \Omega} \frac{m^j M}{1 + |xm^j|} \le 2M.$$

Thus, by Harnack's inequality, there exists a $0 < c < 1$ such that

$$u_j(y) > cu_j(y_0) \qquad \text{for all } j = 1,2,\ldots \quad \text{and} \quad y \in D.$$

This completes the proof of the Claim.

Proof of Theorem 4, *part (ii)*. As the proof is very similar to that of part (i), we just give a sketch of the proof.

Fix m such that $D^c \subset \{|x| < m\}$. Now fix $(x_0, y_0) \in R^d \times R^d$ with $|x_0|, |y_0| \geq m$ and, without loss of generality, assume that $|x_0| \leq |y_0|$. Let j_0 be the positive integer satisfying $m^{j_0} \leq |x_0| < m^{j_0+1}$.

Let

$$\sigma_{j_0} = \sigma_{j_0}(X(\cdot)) = \inf\{t \geq 0 : |X(t)| = m^{j_0}\},$$

$$\sigma_j = \sigma_j(X(\cdot)) = \inf\{t > \sigma_{j+1} : |X(t)| = m^j\}, \qquad j = j_0 - 1, j_0 - 2, \ldots, 1.$$

Also let

$$\tau_j = \tau_j(X(\cdot)) = \inf\{t \geq 0 : |X(t)| = m^j\}, \qquad j = 1, 2, \ldots,$$

and define

$$\Sigma_j = \{|x| = m^j\}, \qquad j = 1, 2, \ldots .$$

For $j = 1, 2, \ldots, j_0$, and $|x| \geq m^j$, define the harmonic measure $\mu_x^j(dy) \in \mathcal{P}(\Sigma_j)$ by $\mu_x^j(dy) = P_x(X(\tau_j) \in dy)$. The same proof as in part (i) shows that there exists a $c > 0$, independent of j_0, and for each $j = 1, 2, \ldots, j_0 - 1$, a measure $\nu^j(dy) \in \mathcal{P}(\Sigma_j)$ such that

$$\mu_x^j(dy) \geq c\nu^j(dz), \qquad \text{for } |x| = m^{j+1} \text{ and } j = 1, 2, \ldots, j_0 - 1. \tag{3.2}$$

Now run $X(t)$ and $Y(s)$ independently, as in part (i), starting from x_0 and y_0 respectively up until $t = \sigma_{j_0-1}(X(\cdot))$ and $s = \sigma_{j_0-1}(Y(\cdot))$. By invoking the strong Markov property at $\sigma_{j_0}(X(\cdot))$ and $\sigma_{j_0}(Y(\cdot))$ and using (3.2), it follows that we may couple $X(\cdot)$ and $Y(\cdot)$ after time $t = \sigma_{j_0-1}(X(\cdot))$ and $s = \sigma_{j_0-1}(Y(\cdot))$ with probability c. Continuing as in part (i), we find that by time $t = \sigma_1(X(\cdot))$ and $s = \sigma_1(Y(\cdot))$, the probability that $X(\cdot)$ and $Y(\cdot)$ are coupled is $\sum_{j=0}^{j_0-2} c(1-c)^j$. Thus property 3 of Theorem 2, part (ii), is satisfied with $\epsilon = \sum_{j=j_0-1}^{\infty} c(1-c)^j$ and $n = m^{j_0}$.

Proof of Proposition 2, *part (i)*. Since $Lv = 0$ in D, $v(X(t \wedge \tau_D))$ is a martingale. By the martingale convergence theorem [1], $\lim_{t\to\infty} v(X(t \wedge \tau_D)) = I_{\tau_D < \infty}(X(\cdot))$ a.s. P_x. Since the process is transient, it follows that there exists an $X_0(\cdot) \in \Omega$ such that $\lim_{t\to\infty} |X_0(t)| = \infty$ and

$$\lim_{t\to\infty} v(X_0(t)) = 0. \tag{3.3}$$

Now assume that Proposition 2(i) is false. Then there exists an $\epsilon > 0$ and a sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} |x_n| = \infty$ such that $v(x_n) > \epsilon$ for all n. By (3.3) we may find a sequence $\{y_n\}_{n=1}^{\infty}$ such that $|y_n| = |x_n|$ and

$$\lim_{n\to\infty} v(y_n) = 0. \tag{3.4}$$

Let $m_n = |x_n| = |y_n|$ and scale space by a factor of m_n. That is, let $v_n(x) = v(m_n x)$, $a^{(n)}(x) = a(m_n x)$, $b^{(n)}(x) = m_n b(m_n x)$, and

$$\hat{L}_n = \sum_{i,j=1}^{d} a_{ij}^{(n)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i^{(n)} \frac{\partial}{\partial x_i}.$$

As in the proof of Theorem 4, it follows that $\hat{L}_n v_n = 0$ in R^d and that by Harnack's inequality, there exists a $c > 0$ independent of n such that

$$\sup_{|x|=1} v_n(x) \le c \inf_{|x|=1} v_n(x). \tag{3.5}$$

But

$$v_n\left(\frac{x_n}{m_n}\right) = v(x_n) > \epsilon \quad \text{and} \quad v_n\left(\frac{y_n}{m_n}\right) = v(y_n) \to 0 \qquad \text{as } n \to \infty.$$

This contradicts (3.5).

Part (ii). In part (i) we showed by scaling and Harnack's inequality that if $\lim_{n\to\infty} v(y_n) = 0$, then in fact $\lim_{|x|\to\infty} v(x) = 0$. In the present case, the same technique shows that if $\lim_{n\to\infty} u(x_n) = \infty$ for some sequence $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} |x_n| = \infty$, then $\lim_{|x|\to\infty} u(x) = \infty$.

Part (iii). Assume to the contrary that u is unbounded. Then the technique of parts (i) and (ii) shows that in fact $\lim_{|x|\to\infty} u(x) = \infty$. Consider first the case that u satisfies $Lu = 0$ in R^d. Then for each $x \in R^d$, $u(X(t \wedge \tau_n))$ is a P_x-martingale, where $\tau_n = \inf\{t \ge 0 : |X(t)| = n\}$. Letting $t \to \infty$, we have $u(x) = E_x u(X(\tau_n)) \ge \inf_{|y|=n} u(y)$, and letting $n \to \infty$ gives $u(x) \equiv \infty$. Now consider the case $Lu = 0$ in D. Then for each $x \in D$, $u(X(t \wedge \tau_D \wedge \tau_n))$ is a P_x-martingale. Letting $t \to \infty$ gives

$$u(x) = E_x(X(\tau_D \wedge \tau_n)) \ge \inf_{|y|=n} u(y) P_x(\tau_D > \tau_n).$$

By the transience assumption, $\lim_{n\to\infty} P_x(\tau_D > \tau_n) > 0$. Thus, letting $n \to \infty$ again gives $u(x) \equiv \infty$.

References

1. K. L. Chung, *A Course in Probability Theory*, 2nd edition, Academic Press Inc., Orlando, Florida, 1974.

2. E. B. Dynkin, *Markov Processes*, Vol. II, Springer-Verlag, Berlin, 1965.

3. D. Gilbarg and J. Serrin, *On isolated singularities of solutions of second order elliptic differential equations*, J. Analyse Math. **4** (1956), 309–340.

4. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer-Verlag, Berlin, 1983.

5. D. Griffeath, *Coupling methods for Markov processes*, in *Advances in Mathematics Supplementary Studies: Studies in Probability and Ergodic Theory 2*, Academic Press, New York, 1978, pp. 1–43.

6. W. S. Kendall, *Nonnegative Ricci curvature and the Brownian coupling property*, Stochastics **19** (1986), 111–129.

7. M. V. Safanov, *Harnack inequalities for elliptic equations and Hölder continuity of their solutions*, Zap. Nauk Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOM I) **96** (1980), 272–287 (Russian).

8. J. Serrin, *On the Harnack inequality for linear elliptic equations*, J. Analyse Math. **4** (1956), 292–308.